

# SOME BIRATIONALITY CRITERIA ON 3-FOLDS WITH $p_g > 1$

MENG CHEN

ABSTRACT. We give some birationality criteria for  $\varphi_m$  ( $m = 4, 5, 6, 7$ ) on general type 3-folds with  $p_g \geq 2$  by means of an intensive classification.

## 1. Introduction

We work over the complex number field  $\mathbb{C}$ .

Pluricanonical maps are usually important tools to study birational geometry of projective varieties. Recently, due to the boundedness theorem separately proved by Hacon-McKernan, Takayama and Tsuji, it has raised a hope to look into explicit birational geometry of high dimensional varieties of general type. In dimension 3, the development is much favorable by virtue of [5, 6] where the following is known:

- ◇ the volume  $\text{Vol}(V) \geq \frac{1}{2660}$ , and
- ◇ the pluricanonical map  $\varphi_m$  is birational for all  $m \geq 73$

where  $V$  is any nonsingular projective 3-fold of general type. Even though, birational geometry in dimension 3 is far from being well-understood.

As far as we know 3-folds with very small volume and very bad pluricanonical behaviors all have invariants  $p_g = q = 0$  and they correspond to surfaces with  $p_g = q = 0$  (i.e. Godeaux surfaces, Campedelli surfaces, Burniet surfaces and so on). Threefolds with  $p_g = 1$  form a very typical class of which pluricanonical behaviors are slightly better. Those with  $p_g > 1$  should be regarded as general objects from the point of view of “moduli”. A feasible strategy to study 3-folds of general type might be to dispart the set of target 3-folds into 3 subsets and to treat each of them by an appropriate method, say,

$$\mathfrak{V}_3 = \{X_1 | p_g(X_1) = 0\} \cup \{X_2 | p_g(X_2) = 1\} \cup \{X_3 | p_g(X_3) \geq 2\}$$

$(=\mathfrak{V}_{3,0})$ 
 $(=\mathfrak{V}_{3,1})$ 
 $(=\mathfrak{V}_{3,2})$

where  $X_i$  denotes an arbitrary 3-fold of general type. Though all above 3 parts are known up to some extent, none of them is clear enough. It is with this motivation that we would like to go further to do a possible classification.

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In this paper we are interested in the third part  $\mathfrak{V}_{3,2}$ . Based on our previous papers [7, 8] and Chen-Zhang [9], we would like to investigate certain parallel phenomenon between surfaces and 3-folds. First of all let us make the following interesting comparison of known results:

Birationality of $\phi_m$ on surfaces $S$ (Bombieri [3], Miyaoka [17])	Birationality of $\varphi_m$ on 3-folds $X$ with $p_g \geq 2$ (Chen [7, 8], Chen-Zhang [9])
$\phi_5$ is birational	$\varphi_8$ is birational $\varphi_7$ (?)
$(K^2, p_g) \neq (1, 2) \iff$ $\phi_4$ is birational	$p_g \geq 3 \implies \varphi_6$ is birational; $p_g = 2$ (?)
$(K^2, p_g) \neq (1, 2), (2, 3) \iff$ $\phi_3$ is birational	$p_g \geq 4 \implies \varphi_5$ is birational; $p_g = 2, 3$ (?)
when $K^2 \geq 10$ or $p_g \geq 6$ , $\phi_2$ is non-birational iff $S$ admits a family of genus 2 curves;  when $K^2 \leq 9$ , $\phi_2$ (?)	when $p_g \geq 5$ , $\varphi_4$ is non-birational iff $X$ admits a genus 2 curve family of canonical degree 1  when $p_g \leq 4$ , $\varphi_4$ (?)
$\phi_1$ (?)	$\varphi_1, \varphi_2, \varphi_3$ (?)

where “?” means an open status,  $\phi_n := \Phi_{|mK_S|}$  and  $\varphi_m := \Phi_{|mK_X|}$ .

We start with a translation of Bombieri’s famous theorem ([3]) from another point of view.

**Theorem 0.** *Let  $S$  be a minimal projective surface of general type. Then  $\varphi_4$  is non-birational if and only if  $S$  admits a genus 2 curve family  $\mathcal{C}$  of canonical degree 1.*

*Proof.* For a general curve  $C \in \mathcal{C}$  with  $(K_S \cdot C) = 1$ , we must have  $C^2 > 0$  since it is an odd number and  $C$  is moving. Then we get  $K_S^2 = 1$  by the Hodge index theorem. The Noether inequality implies  $p_g(S) = 0, 1, 2$ . The case  $p_g(S) = 0$  is impossible since  $(K_S \cdot C) \geq 2$  by Miyaoka [17, Lemma 5]. The case  $p_g(S) = 1$  is impossible either since, otherwise,  $K_S \equiv C$  which contradicts to the fact that  $S$  is simply connected (see Bombieri [3]). Thus  $S$  is a  $(1, 2)$  surface and  $\varphi_4$  is non-birational according to Bombieri [3].

Conversely, since  $\varphi_4$  is non-birational,  $S$  is a  $(1, 2)$  surface by Bombieri [3] and the canonical curve family on  $S$  is the desired family of canonical degree 1.  $\square$

The aim of this paper is to study those open cases on 3-folds. Here is one of the main results:

**Theorem 1.1.** *Let  $X$  be a minimal projective 3-fold of general type and  $p_g(X) \geq 2$ . Then*

- (1)  $\varphi_7$  is birational if and only if either  $p_g(X) > 2$  or  $X$  does not admit any genus 2 curve family  $\mathcal{C}$  of canonical degree  $\frac{2}{3}$ . (see Definition 2.4 for exact meaning of a “curve family”)
- (2)  $\varphi_6$  is birational unless  $p_g = 2$  and  $K_X^3 \leq 6$ .
- (3)  $\varphi_5$  is birational unless  $X$  has one of the following numerical types:
  - 3.1.  $p_g(X) = 3$  and  $K_X^3 \leq 6$ ;
  - 3.2.  $p_g(X) = 2$  and  $K_X^3 \leq 81$ .

We provide some supporting examples which show that all above exceptional cases do really occur.

**Example A.** (1) The general hypersurface  $X := X_{16} \subset \mathbb{P}(1, 1, 2, 3, 8)$  (see Fletcher [11, p151]) of degree 16 has the invariants  $p_g = 2$  and  $K_X^3 = \frac{1}{3}$ . It is clear that  $\varphi_7$  is non-birational. Automatically  $\varphi_6, \varphi_5$  are non-birational either since  $p_g > 0$ . According to our earlier result in [8],  $X$  is canonically fibred by (1,2) surfaces and the relative canonical map of  $\varphi_1$  of  $X$  gives a genus 2 curve family of canonical degree  $\frac{2}{3}$ .

(2) The general hypersurface  $X := X_{14} \subset \mathbb{P}(1, 1, 2, 2, 7)$  of degree 14 has the invariants  $p_g = 2$  and  $K_X^3 = \frac{1}{2}$  and there are 7 orbifold points  $\frac{1}{2}(1, -1, 1)$  on  $X$ . Clearly  $\varphi_7$  is birational. Since the Cartier index of  $X$  is 2, it does not admit any curve family of canonical degree  $\frac{2}{3}$ , which accounts for the birationality of  $\varphi_7$  by virtue of Theorem 1.1(1). Note that  $\varphi_6$  of  $X$  is non-birational.

(3) The general hypersurface  $X := X_{12} \subset \mathbb{P}(1, 1, 1, 2, 6)$  of degree 12 has the invariants  $p_g = 3$  and  $K_X^3 = 1$ , but  $\varphi_5$  is non-birational.

(4) The codimension 2 complete intersection

$$X := X_{4,12} \subset \mathbb{P}(1, 1, 2, 2, 3, 6)$$

of bi-degree (4, 12) has the invariants  $p_g = 2$  and  $K_X^3 = \frac{2}{3}$ , but  $\varphi_5$  is non-birational.

Going on the story in Chen-Zhang [9], we shall characterize the birationality of  $\varphi_4$  as follows:

**Theorem 1.2.** *Let  $X$  be a minimal projective 3-fold of general type and  $X$  satisfies one of the following conditions:*

- 0.  $p_g(X) \geq 5$ . (established in [9])
- 1.  $p_g(X) = 4$  and  $K_X^3 > \frac{72}{5}$ .
- 2.  $p_g(X) = 3$  and  $K_X^3 > 180$ .
- 3.  $p_g(X) = 2$  and  $K_X^3 > 855$ .

*Then  $\varphi_4$  is birational if and only if  $X$  does not admit any genus two curve family  $\mathcal{C}$  of canonical degree 1.*

Our classification in this paper has provided a broader way to find non-trivial examples with  $\varphi_4$  non-birational. See, for instance, the following:

**Example B.** According to Theorem 5.1, any minimal 3-fold  $X$  having terminal singularities,  $p_g(X) = 4$  and  $K_X^3 = 2$  (such that  $\varphi_1$  is generically finite) must have non-birational  $\varphi_4$ . The hypersurface  $X_{10} \subset \mathbb{P}(1, 1, 1, 1, 5)$  is a very special candidate which is smooth.

**Remark 1.** Theorem 1.2 is parallel to Theorem 0. Some other examples with non-birational  $\varphi_4$  can be found in Fletcher [11] and Chen-Zhang [9].

**Remark 2.** The  $\varphi_4$  and  $\varphi_3$  have ever been partially studied by Zhou [24] and Zhu [25] under extra conditions.

At least the following pieces are among the main observations of this paper:

- ◇ For a nef and big  $\mathbb{Q}$ -divisor  $\mathcal{L}$  on a smooth projective surface  $S$  with  $p_g(S) = 1$ , the geometric nature of the linear system  $|K_S + [\mathcal{L}]|$  is difficult to detect, especially when (up to numerical equivalence)  $\mathcal{L} < 2\sigma^*(K_{F_0})$ . Our solution is to deform it into a successful application of Masek’s interesting theorem in [14] – a generalized form of Ein-Lazarsfeld’s argument in [10].
- ◇ When  $X$  is fibred by surfaces with very small invariant  $c_1^2$ , a large ratio  $K_X^3/3c_1^2$  will be much more effective in improving our “canonical restriction inequality” in [9, Lemma 3.7], which will amend whatever we didn’t realize before, but one needs to assume  $K_X^3$  to be large enough.
- ◇ Theorem 0 and the main statements of this paper force us to believe that the existence of certain curve family with very small canonical degree essentially affects the birational geometry of varieties in question.
- ◇ Parallel to the surface case, it is impossible to handle things in a uniform way to treat 3-folds with very small invariants (for instance, with small  $p_g$  and  $K^3$ ). Sometimes the very refined classification of surfaces are needed (see, for example, Claim 5.2.4 and Remark 5.4). It is even inevitable to ask lots of new questions (on surfaces with  $p_g \leq 1$ ) which, unfortunately, are still mysterious to surface experts.

We are in favor of the following symbols:

“ $\sim$ ” denotes linear equivalence or  $\mathbb{Q}$ -linear equivalence;

“ $\equiv$ ” denotes numerical equivalence;

“ $A \geq_{\text{num}} B$ ” means that  $A - B$  is numerically equivalent to an effective  $\mathbb{Q}$ -divisor.

## 2. Definitions, lemmas, notations and the setting

Throughout  $X$  will be a minimal projective 3-fold of general type, on which  $\omega_X = \mathcal{O}_X(K_X)$  is the canonical sheaf and  $K_X$  a canonical divisor.

### 2.1. Fixed notation and setting.

Assume  $p_g(X) := h^0(X, \omega_X) \geq 2$ . We may study the geometry induced from the canonical map  $\varphi_1 : X \dashrightarrow \mathbb{P}^{p_g-1}$  where  $\varphi_1$  is usually a rational map.

Fix an effective Weil divisor  $K_1 \sim K_X$ . Take successive blow-ups  $\pi : X' \rightarrow X$ , which exists by Hironaka's big theorem, such that:

- (i)  $X'$  is nonsingular,
- (ii) the movable part of  $|K_{X'}|$  is base point free,
- (iii) the support of  $\pi^*(K_1)$  is of simple normal crossings.

Denote by  $g$  the composition  $\varphi_1 \circ \pi$ . So  $g : X' \rightarrow \Sigma \subseteq \mathbb{P}^{p_g(X)-1}$  is a morphism. Let  $X' \xrightarrow{f} \Gamma \xrightarrow{s} \Sigma$  be the Stein factorization of  $g$ . We get the following commutative diagram:

$$\begin{array}{ccc}
 X' & \xrightarrow{f} & \Gamma \\
 \pi \downarrow & \searrow g & \downarrow s \\
 X & \xrightarrow{\varphi_1} & \Sigma
 \end{array}$$

We may write  $K_{X'} = \pi^*(K_X) + E_\pi = M_1 + Z_1$ , where  $|M_1|$  is the moving part of  $|K_{X'}|$ ,  $Z_1$  the fixed part and  $E_\pi$  an effective  $\mathbb{Q}$ -divisor which is a sum of distinct exceptional divisors with rational coefficients. For any positive integer  $m$ , whenever taking the round-up of  $m\pi^*(K_X)$ , we always have  $\lceil m\pi^*(K_X) \rceil \leq mK_{X'}$  by the definition of  $\pi^*$ . Since  $h^0(X', \mathcal{O}_{X'}(M_1)) = h^0(\omega_X)$ , we may also write  $\pi^*(K_X) = \underline{M_1 + E'_1}$  where  $E'_1 = Z_1 - E_\pi$  is an effective  $\mathbb{Q}$ -divisor. Set  $d_1 := \dim \varphi_1(X)$ . Clearly one has  $0 < d_1 \leq 3$ .

If  $d_1 = 2$ , a general fiber  $C$  of  $f$  is a smooth projective curve of genus  $\geq 2$ . We say that  $X$  is *canonically fibred by curves*.

If  $d_1 = 1$ , a general fiber  $F$  of  $f$  is a smooth projective surface of general type. We say that  $X$  is *canonically fibred by surfaces* with invariants  $(c_1^2(F_0), p_g(F))$ , where  $F_0$  is the minimal model of  $F$  obtained from the contraction morphism  $\sigma : F \rightarrow F_0$ . We may write  $M \equiv pF$  where  $p \geq p_g(X) - 1$ . Denote  $b := g(\Gamma)$ .

A *generic irreducible element*  $S$  of  $|M|$  means either a general member of  $|M|$  in the case  $d_1 \geq 2$  or, otherwise, a general fiber  $F$  of  $f$ .

For any integer  $m > 0$ ,  $|M_m|$  denotes the moving part of  $|mK_{X'}|$ .

### 2.2. Technical preparation.

We always refer to Chen-Zhang [9, Section 3] for birationality principles [9, Lemma 3.1, Lemma 3.2], the key technical theorem [9, Theorem 3.6] and the “canonical restriction inequality” [9, Lemma 3.7].

### 2.3. Other necessary notions and lemmas.

**Definition 2.1.** Let  $|M|$  be a movable linear system on a normal projective variety  $Z$ . We say that the rational map  $\Phi_{|M|}$  *distinguishes sub-varieties*  $W_1, W_2 \subset Z$  if, set theoretically,  $\overline{\Phi_{|M|}(W_1)} \not\subseteq \overline{\Phi_{|M|}(W_2)}$  and  $\overline{\Phi_{|M|}(W_2)} \not\subseteq \overline{\Phi_{|M|}(W_1)}$ . We say  $\Phi_{|M|}$  *separates points*  $P, Q \in Z$  if  $P, Q \notin \text{Bs}|M|$  and  $\Phi_{|M|}(P) \neq \Phi_{|M|}(Q)$ .

Since all 3-folds considered here have  $p_g(X) > 0$ , we immediately have the following fact which will be tacitly used throughout the paper:

**Fact 2.2.** Under the setting of 2.1,  $\varphi_m := \Phi_{|mK_{X'}|}$  distinguishes different generic irreducible elements  $S$  of  $|M_1|$  for all  $m \geq 2$ .

We will frequently use the following base point freeness due to [3, 4, 12, 21]:

**Fact 2.3.** Let  $S_0$  be a minimal projective surface of general type and  $p_g(S_0) > 0$ . Then  $|2K_{S_0}|$  is base point free.

**Definition 2.4.** Let  $Z$  be a normal projective  $\mathbb{Q}$ -Gorenstein variety. Let  $\theta : Z' \rightarrow Z$  be a birational morphism and  $h : Z' \rightarrow W$  be a fibration onto another normal variety  $W$  with  $\dim(W) = \dim(Z) - 1$ . Then we call  $\mathcal{C} := \{\theta(F) | F \text{ is a fiber of } h\}$  a *curve family* on  $Z$ . For a general member  $C \in \mathcal{C}$ , the rational number  $\deg(\mathcal{C}) := (K_Z \cdot C)$  is referred to as *the canonical degree of  $\mathcal{C}$* .

Note that  $\deg(\mathcal{C})$  is independent of the birational morphism  $\theta$  by the intersection theory.

**Definition 2.5.** Let  $S$  be a nonsingular projective surface. For a point  $P \in S$ ,  $P$  is said to be *very general* if  $P$  lies in the complement of the union of countable curves on  $S$ .

**Lemma 2.6.** Let  $S$  be a nonsingular projective surface. Let  $\mathcal{L}$  be a nef and big  $\mathbb{Q}$ -divisor on  $S$  satisfying the following conditions:

- (1)  $\mathcal{L}^2 > 8$ .
- (2)  $(\mathcal{L} \cdot C_P) \geq 4$  for all irreducible curves  $C_P$  passing through any very general point  $P \in S$ .

Then the linear system  $|K_S + \lceil \mathcal{L} \rceil|$  separates two distinct points in very general positions. Consequently it gives a birational map.

*Proof.* This is a direct result from the proof of Masek [14, Proposition 4]. We keep the same notation there. Let  $p, q$  be two distinct very general points on  $S$ . Then we are in the situation  $\mu_p = \mu_q = 0$ . Just set  $\beta_{1,p} = \beta_{1,q} = 2$  and  $\beta_{2,p} = \beta_{2,q} = 2$ . Then our situation here fits into all numerical requirements there and, as a result, the proof follows. Note, however, Masek's condition of "M being ample" is set to secure the local positivity at every points in order to obtain base point freeness and very ampleness. To obtain birationality, the "nef and big" condition is sufficient.  $\square$

**Lemma 2.7.** *Let  $\pi : X' \rightarrow X$  be a birational morphism from a nonsingular model  $X'$  onto  $X$  which is a minimal projective 3-fold of general type. Assume  $f : X' \rightarrow \mathbb{P}^1$  be a fibration with the general fiber  $F$ . Let  $\sigma : F \rightarrow F_0$  be the birational contraction onto the minimal model. Set  $\tau_0 := \frac{K_X^3}{3K_{F_0}^2}$ . Then*

- (i) *For any rational number  $\delta > 0$ , there are two positive integers  $N$  and  $n$  such that  $\frac{n}{N} = \tau_0 - \delta$  and that*

$$N\pi^*(K_X) \geq_{\text{num}} nF.$$

- (ii) *For any small rational number  $\varepsilon_0$ , there exists an effective  $\mathbb{Q}$ -divisor  $J_{\varepsilon_0}$  such that*

$$\pi^*(K_X)|_F \equiv \left(\frac{\tau_0}{\tau_0 + 1} - \varepsilon_0\right)\sigma^*(K_{F_0}) + J_{\varepsilon_0}.$$

*Proof.* (1) For any sufficiently large and divisible integer  $m > 0$  (such that  $m$  is divisible by the Cartier index of  $X$ ), the Riemann-Roch on  $X'$  implies

$$P_m(X') = h^0(X', m\pi^*(K_X)) \approx \frac{1}{6}K_X^3 m^3.$$

On the other hand, the Riemann-Roch on  $F$  gives

$$P_m(F) \approx \frac{1}{2}K_{F_0}^2 m^2.$$

Therefore  $P_m(X') > (\tau_0 - \delta)mP_m(F)$  for  $m \gg 0$ . Consider the restriction maps:

$$H^0(X', M_m - tF) \xrightarrow{\theta_t} V_{m,t} \subset H^0(F, mK_F)$$

where  $t \geq 0$  and  $V_{m,t}$  is the image vector space. Since  $\dim(V_{m,t}) \leq P_m(F)$  for all  $t$ , we have

$$m\pi^*(K_X) - (\tau_0 - \delta)mF \geq M_m - (\tau_0 - \delta)mF > 0$$

for all large and divisible integers  $m$  (such that  $(\tau_0 - \delta)m$  is integral). Pick a large such integer  $m = l_0$  and set  $N := l_0$  while  $n := (\tau_0 - \delta)l_0$ . So we get (i).

(2) Statement (ii) follows directly from our canonical restriction inequality [9, Lemma 3.7] since  $\frac{p}{m_0} \mapsto \tau_0$  in this situation. We are done.  $\square$

**Lemma 2.8.** *Let  $S$  be a nonsingular projective surface on which there is a nef and big  $\mathbb{Q}$ -divisor  $L$  and a smooth curve  $C$  with  $(L \cdot C) > 0$ . Set  $\nu_0 := \frac{L^2}{2(L \cdot C)}$ . Then*

$$L \geq_{\text{num}} (\nu_0 - \delta_0)C$$

*for all very small rational numbers  $\delta_0 > 0$ .*

*Proof.* This is similar to Lemma 2.7. For a very large and divisible integer  $m$ , we have

$$h^0(S, mL) \approx \frac{1}{2}L^2m^2$$

and  $h^0(C, mL|_C) \approx (L \cdot C)m$ . Then the statement follows by simply considering the restriction maps:

$$H^0(S, mL) \longrightarrow H^0(C, mL|_C).$$

We omit other details.  $\square$

The following result is also frequently used to distinguish curves on surfaces.

**Lemma 2.9.** *Let  $\pi : X' \rightarrow X$  be a birational morphism from nonsingular projective 3-fold  $X'$  onto a minimal 3-fold  $X$  of general type. Let  $f : X' \rightarrow \mathbb{P}^1$  be a fibration. Assume  $\mathcal{O}(p) \hookrightarrow f_*\omega_{X'}$  for some integer  $p > 0$ . Then, for all  $t > 0$  with  $|tp\sigma^*(K_{F_0})|$  base point free, one has*

$$t(p+2)\pi^*(K_X)|_F \geq tp\sigma^*(K_{F_0})$$

where  $F$  is a general fiber of  $f$  and  $\sigma : F \rightarrow F_0$  is the contraction onto the minimal model.

*Proof.* By assumption, one has

$$f_*\omega_{X'/\mathbb{P}^1}^{\otimes tp} \hookrightarrow f_*\omega_{X'}^{\otimes t(p+2)}.$$

By the semi-positivity theorem for  $f_*\omega_{X'/\mathbb{P}^1}^{\otimes tp}$ , we see that  $f_*\omega_{X'/\mathbb{P}^1}^{\otimes tp}$  is generated by global sections. Thus the local sections along the general fiber  $F$  can be glued into some global sections of  $f_*\omega_{X'}^{\otimes t(p+2)}$ . This implies

$$\begin{aligned} |t(p+2)K_{X'}|_F &\supset |tp\sigma^*(K_{F_0})|_{\text{mov}} + (\text{fixed divisor}) \\ &= |tp\sigma^*(K_{F_0})| + (\text{fixed divisor}) \end{aligned}$$

since  $|tp\sigma^*(K_{F_0})|$  is base point free. Or, in divisor language, one has

$$t(p+2)\pi^*(K_X)|_F \geq M_{t(p+2)}|_F \geq tp\sigma^*(K_{F_0})$$

where  $|M_{t(p+2)}|$  is assumed to be the moving part of  $|t(p+2)K_{X'}|$ .  $\square$

### 3. Proof of Theorem 1.1 (Part 1)

#### 3.1. Characterization of the birationality of $\varphi_7$ .

We start with the proof of Theorem 1.1(1). Since  $p_g(X) \geq 2$ , we have an induced fibration  $f : X' \rightarrow \Gamma$ .

Assume  $\varphi_7$  is non-birational. Then, by [7, Theorem 1.2 and Section 4], one has  $p_g(X) = 2$ ,  $\Gamma \cong \mathbb{P}^1$  and a general fiber  $F$  of  $f$  is a (1,2) surface. We show the existence of the curve family  $\mathcal{C}$  as claimed in the statement. Pick a general fiber  $F$  and take  $|G|$  to be the moving part of  $|K_F|$ . Take further necessary birational modifications to  $\pi$  such that (and thus may assume) the relative canonical map of  $f$  is a morphism.



By the surface theory (see [2]), a generic irreducible element  $C$  of  $|G|$  is a smooth curve of genus 2. We have  $m_0 = 1$ ,  $p = 1$  and [9, Lemma 3.7] implies  $\beta \mapsto \frac{1}{2}$ . We have already known  $\xi := (\pi^*(K_X) \cdot C) \geq \frac{3}{5}$  in [7, Section 4]. Take  $m = 6$ . Since  $\alpha_6 \geq 2\xi > 1$ , [9, Theorem 3.6] implies  $\xi \geq \frac{2}{3}$ . Now  $\xi > \frac{2}{3}$  is impossible since, otherwise,  $\alpha_7 \geq 3\xi > 2$  and  $\varphi_7$  will be birational by [9, Theorem 3.6], a contradiction. Thus  $\xi = \frac{2}{3}$ . Take  $\mathcal{C} := \{\pi(C_t) | C_t \text{ is a fibre of the relative canonical map of } f\}$ . Clearly  $\deg(\mathcal{C}) = \xi = \frac{2}{3}$  by the projection formula.

Conversely, assume  $p_g(X) = 2$  and there is a genus 2 curve family  $\mathcal{C}$  of canonical degree  $\frac{2}{3}$ . We study the general fiber  $F$  of the canonically induced fibration  $f : X' \rightarrow \Gamma$ .

**Claim 3.1.1.**  $\Gamma \cong \mathbb{P}^1$  and  $F$  must be a (1,2) surface.

*Proof.* Modulo further birational modifications, we may assume that the curve family  $\mathcal{C}$  is free on  $X'$ . Pick a general curve  $\hat{C} \subset X'$  such that  $\pi(\hat{C})$  dominates a general curve in  $\mathcal{C}$  on  $X$ . First, we see  $\hat{C} \subset F$  for some general fiber  $F$ . Otherwise,  $f(\hat{C}) = \Gamma$ ,  $h^0(\hat{C}, F|_{\hat{C}}) \geq 2$  and then  $(\pi^*(K_X) \cdot \hat{C}) \geq \deg(F|_{\hat{C}}) \geq 2$  which contradicts to the assumption. Secondly, if  $g(\Gamma) > 0$ , then we have  $\pi^*(K_X)|_F \cong \sigma^*(K_{F_0})$  by [9, Lemma 4.7] and

$$(\pi^*(K_X) \cdot \hat{C}) = (\pi^*(K_X)|_F \cdot \hat{C}) = (\sigma^*(K_{F_0}) \cdot \hat{C}) \geq 1$$

since  $\hat{C}$  is moving in a family on  $F$ , which is again a contradiction. Thus we have seen  $\Gamma \cong \mathbb{P}^1$ .

Now we study the numerical type of the general fiber  $F$ . We have

$$\frac{2}{3} = (\pi^*(K_X)|_F \cdot \hat{C}) \geq \frac{1}{2}(\sigma^*(K_{F_0}) \cdot \hat{C}),$$

which implies  $(\sigma^*(K_{F_0}) \cdot \hat{C}) = 1$ . Observing that  $\hat{C}$  is moving, the Hodge index theorem and the assumption  $g(\hat{C}) = 2$  imply  $K_{F_0}^2 = 1$ . By the surface theory and the proof of [8, Claim 2.14],  $F$  must be a (1,2) surface and  $\hat{C}$  is the moving part of  $|K_F|$ . So we have  $\hat{C} = C$  as a general member of  $|G|$  on  $F$ .  $\square$

**Claim 3.1.2.**  $|M_7||_C = |2K_C|$ .

*Proof.* If  $C$  is a general curve in the moving part of  $|K_F|$ , one has  $K_F = \sigma^*(K_{F_0}) + E_{(0)}$  and  $E_{(0)} \cap C$  is a single point  $P \in C$  with  $2P \sim K_C$ , which is due to the fact that  $|K_{F_0}|$  has exactly one base point. In particular, we have  $K_F|_C = 2P$ . This means that  $(\pi^*(K_X)|_F + E_\pi|_F)|_C = K_F|_C = 2P$ , once we fix a general fiber  $F$  and a general curve  $C$  on  $F$ .

Since  $\mathcal{O}_\Gamma(1) \hookrightarrow f_*\omega_{X'}$ , Lemma 2.9 implies  $3\pi^*(K_X)|_F \geq C$ . Now the Kawamata-Viehweg vanishing theorem ([13, 22]) gives

$$\begin{aligned} |K_{X'} + [5\pi^*(K_X)] + F|_F &= |K_F + [5\pi^*(K_X)]|_F| \\ &\supset |K_F + [2\pi^*(K_X)|_F] + C| + (\text{fixed divisor}) \end{aligned}$$

and

$$|K_F + \lceil 2\pi^*(K_X)|_F \rceil + C|_C = |K_C + D|$$

with  $\deg(D) \geq 2\xi \geq \frac{4}{3}$ . Since  $\pi^*(K_X)|_F \leq K_F$ , we have  $D = 2P$ .

Noting that  $|K_{X'} + \lceil 5\pi^*(K_X) \rceil + F| + (\text{fixed divisor}) \subset |7K_{X'}|$  and  $|K_C + D|$  is movable, we have

$$|M_7|_C \supset |M_7|_C \supset |K_C + D| + (\text{fixed divisor}).$$

Since  $\deg(M_7|_C) \leq 7(\pi^*(K_X)|_F \cdot C) = \frac{14}{3}$ , we get  $\deg(M_7|_C) \leq 4$ . Thus the only possibility is  $M_7|_C \sim K_C + D$  and  $|M_7|_C = |K_C + D| = |2K_C|$ , which gives a finite map of degree 2.  $\square$

Clearly  $\varphi_7|_F$  distinguishes different general curves  $C$ , we see  $\varphi_7$  is generically finite of degree 2. So we conclude Theorem 1.1(1).

### 3.2. Birationality of $\varphi_6$ .

In this part of the text we shall prove Theorem 1.1(2).

By [7, Theorem 1.2, Theorem 3.3], we only need to assume  $p_g(X) = 2$  and  $\Gamma \cong \mathbb{P}^1$  to prove the birationality of  $\varphi_6$ .

We have an induced fibration  $f : X' \rightarrow \Gamma$  where we pick a general fiber  $F$ . By the surface theory,  $F$  must be among the following types, since  $p_g(F) > 0$ :

- a.  $K_{F_0}^2 \geq 3$ .
- b.  $K_{F_0}^2 = 2$ .
- c.  $K_{F_0}^2 = 1$ ,  $p_g(F) = 1$ .
- d.  $K_{F_0}^2 = 1$ ,  $p_g(F) = 2$ .

Clearly it is sufficient to prove the birationality of  $\varphi_6|_F$  for a general fiber  $F$ .

**Claim 3.2.1.** *If  $F$  is of Type (a),  $\varphi_6$  is birational.*

*Proof.* By Kawamata-Viehweg vanishing, we have

$$\begin{aligned} |K_{X'} + \lceil 4\pi^*(K_X) \rceil + M_1|_F &= |K_F + \lceil 4\pi^*(K_X) \rceil|_F \\ &\supset |K_F + \lceil L_4 \rceil| + (\text{fixed divisor}) \end{aligned} \quad (3.1)$$

where  $L_4 := 4\pi^*(K_X)|_F$  is a nef and big  $\mathbb{Q}$ -divisor.

By [9, Lemma 3.7], there exists an effective  $\mathbb{Q}$ -divisor  $H_\varepsilon$  such that

$$\pi^*(K_X)|_F \equiv \left(\frac{1}{2} - \varepsilon\right)\sigma^*(K_{F_0}) + H_\varepsilon \quad (3.2)$$

for all very small rational numbers  $\varepsilon > 0$ . Noting that both  $\pi^*(K_X)|_F$  and  $\sigma^*(K_{F_0})$  are nef, we get

$$L_4^2 \geq 16 \cdot \left(\frac{1}{2} - \varepsilon\right)^2 \cdot \sigma^*(K_{F_0})^2 \geq 3(2 - 4\varepsilon)^2 > 8 \quad (3.3)$$

whenever  $\varepsilon$  is small enough.

( $\dagger$ ) For a very general point  $P \in F$ , any curve  $C_P$  passing through  $P$  is of general type, i.e.  $g(C_P) \geq 2$ . Then an easy exercise will show  $(\sigma^*(K_{F_0}) \cdot C_P) \geq 2$  simply due to the fact  $K_{F_0}^2 > 1$ . Find a sequence of

small rational numbers  $\{\varepsilon_n\}$  and by the choice of  $P$ , we may assume  $C_P \notin \bigcup_n H_{\varepsilon_n}$ . In particular,  $(C_P \cdot H_{\varepsilon_n}) \geq 0$  for all  $n > 0$ . Thus

$$(L_4 \cdot C_P) \geq 4\left(\frac{1}{2} - \varepsilon_n\right)(\sigma^*(K_{F_0}) \cdot C_P) = (2 - 4\varepsilon_n)(\sigma^*(K_{F_0}) \cdot C_P).$$

Taking the limit while  $n \mapsto +\infty$ , we have  $(L_4 \cdot C_P) \geq 4$ .

Now Lemma 2.6 implies that  $|K_F + L_4|$  gives a birational map. Thus  $\varphi_6$  is birational.  $\square$

**Claim 3.2.2.** *If  $K_X^3 > 6$  and  $F$  is of Type (b),  $\varphi_6$  is birational.*

*Proof.* Take  $|G| := |2\sigma^*(K_{F_0})|$  if  $p_g(F) \neq 3$  and, otherwise, take  $|G|$  to be the moving part of  $|K_F|$ . Clearly  $\varphi_6|_F$  distinguishes different general irreducible elements of  $|G|$  by Lemma 2.9 and Relation (3.1) respectively. Unfortunately Lemma 2.6 is no longer effective since  $L_4^2 \leq 8$ .

When  $K_X^3 > 6$ ,  $\tau_0 := \frac{K_X^3}{3K_{F_0}^2} > 1$  and Lemma 2.7 implies that there is an effective  $\mathbb{Q}$ -divisor  $J_\varepsilon$  with

$$\pi^*(K_X)|_F \equiv \left(\frac{\tau_0}{\tau_0 + 1} - \varepsilon\right)\sigma^*(K_{F_0}) + J_\varepsilon$$

for any small rational numbers  $\varepsilon > 0$ . Take a small  $\varepsilon_0$  such that  $\eta_0 := \frac{\tau_0}{\tau_0 + 1} - \varepsilon_0 > \frac{1}{2}$ .

If  $p_g(F) \neq 3$ , a generic irreducible element  $C$  of  $|G|$  is even and non-hyperelliptic. We see  $\beta \geq \frac{1}{2}\eta_0 > \frac{1}{4}$ . Since  $\alpha_6 = (6 - 2 - \frac{1}{\beta})\xi > 0$ , [9, Theorem 3.6] implies the birationality of  $\varphi_6$ .

If  $p_g(F) = 3$ ,  $\beta \geq \eta_0 > \frac{1}{2}$  and  $\xi := (\pi^*(K_X)|_F \cdot C) > \frac{1}{2}(\sigma^*(K_{F_0}) \cdot C) \geq 1$ . Since  $\alpha_6 \geq (6 - 2 - \frac{1}{\beta})\xi > 2$ ,  $\varphi_6$  is birational again by [9, Theorem 3.6]. The claim is proved.  $\square$

**Claim 3.2.3.** *If  $K_X^3 > 3$  and  $F$  is of Type (c),  $\varphi_6$  is birational.*

*Proof.* Take  $|G| := |2\sigma^*(K_{F_0})|$ , which is base point free. Again Lemma 2.9 says  $\varphi_6|_F$  distinguishes different general members in  $|G|$ .

Since  $\tau_0 = \frac{K_X^3}{3K_{F_0}^2} > 1$ , we have  $\beta > \frac{1}{4}$  similarly. Thus we still have  $\alpha_6 > 0$  and  $\varphi_6$  is birational by [9, Theorem 3.6].  $\square$

**Claim 3.2.4.** *If  $K_X^3 > 5$  and  $F$  is of Type (d), then  $\varphi_6$  is birational.*

*Proof.* Take  $|G|$  to be the moving part of  $|K_F|$ . Then we have  $\xi = (\pi^*(K_X)|_F \cdot C) \geq \frac{2}{3}$  from 3.1.

Since  $\tau_0 = \frac{K_X^3}{3K_{F_0}^2} > \frac{5}{3}$ , we have  $\frac{p}{m_0} > \frac{5}{3}$  and  $\beta > \frac{5}{8}$  by Lemma 2.7. Since  $\alpha_5 \geq (5 - 1 - \frac{3}{5} - \frac{8}{5})\xi > 1$ , we get  $\xi \geq \frac{4}{5}$  by [9, Theorem 3.6]. Now  $\alpha_6 \geq \frac{14}{5}\xi > 2$  which implies the birationality of  $\varphi_6$  by [9, Theorem 3.6] once more.  $\square$

We have proved Theorem 1.1(2).

#### 4. Proof of Theorem 1.1 (Part 2)

In this section we shall work on the birationality of  $\varphi_5$ . By [7, Theorem 1.2], we only need to study the cases  $p_g(X) = 2, 3$ .

##### 4.1. $\varphi_5$ in the case $p_g(X) = 3$ and $d_1 = 2$ .

When  $d_1 = 2$ , a general fiber  $C$  of  $f$  is a curve of genus  $\geq 2$ . Pick a general member  $S \in |M_1|$ . Take  $|G| := |S|_S$  where  $S|_S \equiv eC$  with  $e \geq 1$ . Taking the restriction, we get

$$\pi^*(K_X)|_S \equiv S|_S + E'_1|_S.$$

Set  $L := \pi^*(K_X)|_S$  which is an effective nef and big  $\mathbb{Q}$ -divisor. Clearly, one has  $L^2 \geq \xi$  by definition.

**Claim 4.1.1.** When  $g(C) \geq 3$ ,  $\xi > 1$ ; when  $g(C) = 2$ ,  $\xi \geq 1$ .

*Proof.* This is a direct consequence of [9, Theorem 3.6]. In fact, we have  $p = 1$  and  $\beta = 1$ . When  $g(C) \geq 3$ , [9, Theorem 3.6] implies  $\xi \geq \frac{4}{3} > 1$ . When  $g(C) = 2$ , clearly one has  $\xi \geq \frac{2}{3}$ . But by repeated optimizations using [9, Theorem 3.6], one has no difficulty to see  $\xi \geq 1$ .  $\square$

**Claim 4.1.2.**  $K_X^3 > 1$  if and only if  $L^2 > 1$ .

*Proof.* To see this, we have the following inequality:

$$K_X^3 = \pi^*(K_X)^3 = (\pi^*(K_X)^2 \cdot S) + (\pi^*(K_X)^2 \cdot E'_1) \geq L^2.$$

On the other hand, if we take a sufficiently large integer  $m$  such that  $|m\pi^*(K_X)|$  is base point free, then a general member  $T$  is a smooth surface and we apply this to estimate  $L^2$  by the Hodge index theorem:

$$\begin{aligned} L^2 &= \frac{1}{m}(\pi^*(K_X)|_T \cdot S|_T) \geq \frac{1}{m} \sqrt{(\pi^*(K_X)|_T)^2 \cdot (S|_T)^2} \\ &\geq \sqrt{K_X^3 \cdot \xi} \geq \sqrt{K_X^3}. \end{aligned}$$

So the claim is true.  $\square$

**Claim 4.1.3.** When  $L^2 > 1$ ,  $|K_S + \lceil 3L \rceil|$  gives a birational map.

*Proof.* In fact, we have  $L = \pi^*(K_X) = C + E'_1|_S$ . There are two cases: (i)  $\xi > 1$ ; (ii)  $\xi = 1$ .

The first case is easier since the vanishing theorem gives

$$|K_S + \lceil 2L \rceil + C|_C = |K_C + \lceil 2L \rceil|_C|$$

and clearly the linear system on the right hand side gives an embedding as  $\deg(\lceil 2L \rceil|_C) \geq 2\xi > 2$ . Thus  $|K_S + 3L|$  gives a birational map.

Assume, from now on,  $\xi = 1$ . Then  $L^2 > 1$  implies  $(L \cdot E'_1|_S) > 0$ . Let us consider the Zariski decomposition of the effective  $\mathbb{Q}$ -divisor  $\hat{J} := L + E'_1|_S$ . Since  $L$  is nef and  $E'_1|_S$  is effective, we may write  $E'_1|_S = N^+ + N^-$  such that:

- (1)  $L + N^+$  is nef;

- (2)  $(L + N^+) \cdot N^- = 0$ ;
- (3) both  $N^+$  and  $N^-$  are effective  $\mathbb{Q}$ -divisors.

Clearly,  $(L \cdot N^+) > 0$ . We want to show  $(C \cdot N^+) > 0$ . In fact, if  $N^- \equiv 0$ , then we have  $(C \cdot N^+) = \xi > 0$ . Otherwise, we may always assume  $N^- \not\equiv 0$ , which means  $N^{-2} < 0$  since  $L + N^+$  is nef and big. From the property (2), we see  $(N^+ \cdot N^-) = -(L \cdot N^-) \leq 0$  since  $L$  is nef. Since  $L = C + N^+ + N^-$ , so  $(L \cdot N^+) > 0$  implies  $(C \cdot N^+) + N^{+2} > 0$ . Suppose  $(C \cdot N^+) = 0$ , then  $N^+$  is vertical with respect to the fibration on  $S$  and  $N^{+2} \leq 0$ , which contradicts to  $N^{+2} = (C \cdot N^+) + N^{+2} > 0$ . So we have proved  $(C \cdot N^+) > 0$ .

Noting that

$$|K_S + \lceil 2L + N^+ \rceil + C| + (\text{fixed divisor}) \subset |K_S + \lceil 3L \rceil|$$

and that the vanishing theorem gives

$$|K_S + \lceil 2L + N^+ \rceil + C|_C = |K_C + D_2|$$

with  $\deg(D_2) \geq 2\xi + (C \cdot N^+) > 2$ . Thus  $|K_S + \lceil 2L + N^+ \rceil + C|$  gives a birational map and so does  $|K_S + \lceil 3L \rceil|$ . We are done.  $\square$

On the other hand, Kawamata-Viehweg vanishing gives

$$\begin{aligned} |K_{X'} + \lceil 3\pi^*(K_X) \rceil + S|_S &= |K_S + \lceil 3\pi^*(K_X) \rceil|_S \\ &\supset |K_S + \lceil 3L \rceil| + (\text{fixed divisor}) \end{aligned}$$

So we see that  $\varphi_5$  is birational whenever  $K_X^3 > 1$ .

When  $K_X^3 = 1$ , automatically  $L^2 = 1$  and  $\xi = 1$ . Thus  $g(C) = 2$ .

So we have proved the following:

**Theorem 4.1.** *Let  $X$  be a minimal projective 3-fold of general type. Assume  $K_X^3 > 1$ ,  $p_g(X) = 3$  and  $d_1 = 2$ . Then  $\varphi_5$  is birational.*

**Remark 4.2.** Theorem 4.1 is sharp due to Example A (3). It is interesting to know whether  $K_X^3 > 1$  is necessary to get the birationality of  $\varphi_5$ .

#### 4.2. $\varphi_5$ in the case $p_g(X) = 3$ and $d_1 = 1$ .

Pick a general fiber  $F$  of the induced fibration  $f : X' \rightarrow \Gamma$ . By [7, Theorem 3.3], it is sufficient to assume  $b = g(\Gamma) = 0$ , i.e.  $\Gamma \cong \mathbb{P}^1$ . Note that  $p_g(X) > 0$  implies  $p_g(F) > 0$  and thus  $F$  must be among the following types by the surface theory:

- (i)  $(K_{F_0}^2, p_g(F)) = (1, 2)$ ;
- (ii)  $(K_{F_0}^2, p_g(F)) = (2, 3)$ ;
- (iii) other surfaces with  $p_g(F) > 0$ .

It suffices to show  $\varphi_5|_F$  is birational for a general fiber  $F$ . One has  $m_0 = 1$  and  $p = 2$ . By [9, Lemma 3.7], for any rational number  $\varepsilon > 0$ , there is an effective  $\mathbb{Q}$ -divisor  $H_\varepsilon$  such that

$$\pi^*(K_X)|_F \equiv \left(\frac{2}{3} - \varepsilon\right)\sigma^*(K_{F_0}) + H_\varepsilon. \quad (4.1)$$

**Claim 4.2.1.** *If  $F$  is of Type (ii),  $\varphi_5$  is birational.*

*Proof.* Take  $|G|$  to be the moving part of  $|K_F|$ . Then a general member  $C \in |G|$  is a smooth curve of genus 3 and  $|G|$  gives a generically finite map (see [2, p226]). Besides, Relation (4.1) implies  $\beta \geq \frac{2}{3} - \varepsilon$  for any small  $\varepsilon > 0$  and thus  $\xi \geq \frac{2}{3}(\sigma^*(K_{F_0}) \cdot C) \geq \frac{4}{3}$ . Take  $m = 4$  and then  $\alpha_4 := (4 - 1 - \frac{1}{p} - \frac{1}{\beta})\xi > 1$ . By [9, Theorem 3.6], one gets  $\xi \geq \frac{3}{2}$ . Since Kawamata-Viehweg vanishing gives

$$|K_{X'} + \lceil 3\pi^*(K_X) \rceil + F|_F \supset |K_F + \lceil 3\pi^*(K_X)|_F \rceil + (\text{fixed divisor}) \quad (4.2)$$

where the last linear system distinguishes different general curves  $C$  and  $\alpha_5 \geq 2\xi \geq 3$ , [9, Theorem 3.6] implies the birationality of  $\varphi_5$ .  $\square$

**Claim 4.2.2.** *If  $F$  is of Type (iii),  $\varphi_5$  is birational.*

*Proof.* We take  $|G| = |2\sigma^*(K_{F_0})|$ . By the surface theory, we know that  $|G|$  is base point free and a general member  $C$  of  $|G|$  is non-hyperelliptic. Lemma 2.9 implies that  $\varphi_4|_F$  distinguishes different general curves  $C$ . On the other hand, Equation (4.1) implies  $\beta \mapsto \frac{1}{3}$ . So  $\alpha_5 = (5 - 1 - \frac{1}{2} - \frac{1}{\beta})\xi > 0$ . Noting that  $C$  is even and non-hyperelliptic, [9, Theorem 3.6] implies the birationality of  $\varphi_5$ .  $\square$

**Claim 4.2.3.** *If  $F$  is of Type (i) and  $K_X^3 > 6$ , then  $\varphi_5$  is birational.*

*Proof.* On  $F$ , take  $|G|$  to be the moving part of  $|K_F|$ . Since  $K_X^3 > 6$ , we have  $\tau_0 > 2$  and Lemma 2.7 implies  $\beta > \frac{2}{3}$ . Similar to the case with  $F$  being of type (ii), it suffices to study  $\varphi_5|_C$ . Recall we have  $m_0 = 1$  and  $p = 2$ . We have  $\xi = (\pi^*(K_X)|_F \cdot C) \geq \beta \cdot (\sigma^*(K_{F_0}) \cdot C) > \frac{2}{3}$ . By repeatedly running [9, Theorem 3.6], one would get  $\xi \geq 1$ . Now take  $m = 5$ . We see  $\alpha_5 > 2\xi \geq 2$ . Thus, by [9, Theorem 3.6],  $\varphi_5$  is birational.  $\square$

So we can conclude the following:

**Theorem 4.3.** *Let  $X$  be a minimal projective 3-fold of general type. Assume  $K_X^3 > 6$ ,  $p_g(X) = 3$  and  $d_1 = 1$ . Then  $\varphi_5$  is birational.*

#### 4.3. $\varphi_5$ in the case $p_g(X) = 2$ .

Automatically  $d_1 = 1$  and this is parallel to 3.2, but we are studying  $\varphi_5$  instead. We keep the notation there and will omit redundant arguments.

**Claim 4.3.1.** *If  $K_{F_0}^2 \geq 19$ ,  $\varphi_5$  is birational.*

*Proof.* The Bogomolov-Miyaoka-Yau inequality implies

$$p_g(F) = \chi(\mathcal{O}_F) - 1 + q(F) \geq \frac{1}{9}K_{F_0}^2 - 1 > 1.$$

Take  $|G|$  to be the moving part of  $|K_F|$ . Modulo birational modifications, we may assume that  $|G|$  is base point free and so a generic irreducible element  $C$  of  $|G|$  is smooth. We still have Relation (4.2) and set  $L = \pi^*(K_X)|_F$ . The linear system  $|K_F + [3L]|$  clearly distinguishes different general curves  $C$ .

If  $|G|$  is not composed of a pencil, we have  $C^2 \geq 2$ . By (3.2), we have  $\beta \mapsto \frac{1}{2}$ . Since  $\alpha_5 \geq \xi \geq \frac{1}{2}(\sigma^*(K_{F_0}) \cdot C) \geq \frac{1}{2}\sqrt{K_{F_0}^2 \cdot C^2} > 2$ ,  $\varphi_5$  is birational by [9, Theorem 3.6].

If  $|G|$  is composed of a pencil and  $K_{F_0}^2 \geq 19$ , pick a generic irreducible element  $C$ . We study the rational number  $\xi = (L \cdot C)$ . If  $\xi > 2$ , then  $\alpha_5 \geq \xi > 2$  and  $\varphi_5$  is birational. Otherwise, we have  $\xi \leq 2$ . Since  $\nu_0 = \frac{L^2}{2\xi} \geq \frac{19}{16}$ , Lemma 2.8 implies

$$L \geq_{\text{num}} \left(\frac{19}{16} - \delta\right)C$$

for any small rational number  $\delta > 0$ , which means  $\beta \geq \frac{19}{16} - \delta$ . Note that  $\xi \geq \frac{1}{2}(\sigma^*(K_{F_0}) \cdot C) \geq 1$ . Now  $\alpha_5 \geq (5 - 1 - 1 - \frac{1}{\beta})\xi > 2$  and so  $\varphi_5$  is birational by [9, Theorem 3.6] once more.  $\square$

**Claim 4.3.2.** *If  $K_X^3 > 81$  and  $K_{F_0}^2 \leq 18$ ,  $\varphi_5$  is birational.*

*Proof.* We prove the statement by analyzing different numerical types of  $F$ .

**a.** Assume  $3 \leq K_{F_0}^2 \leq 18$ . Then since  $\tau_0 = \frac{K_X^3}{3K_{F_0}^2} > \frac{3}{2}$ , we have

$$\pi^*(K_X) \equiv \frac{3}{2}F + E_{3/2}$$

where  $E_{3/2}$  is an effective  $\mathbb{Q}$ -divisor and

$$\pi^*(K_X)|_F \geq_{\text{num}} \frac{3}{5}\sigma^*(K_{F_0})$$

by Lemma 2.7. By the vanishing theorem, we have

$$\begin{aligned} |K_{X'} + [4\pi^*(K_X) - \frac{2}{3}E_{3/2}]|_F &= |K_F + [4\pi^*(K_X) - F - \frac{2}{3}E_{3/2}]|_F| \\ &\supset |K_F + [Q]| + (\text{fixed divisor}) \end{aligned}$$

where

$$Q := (4\pi^*(K_X) - F - \frac{2}{3}E_{3/2})|_F \equiv \frac{10}{3}\pi^*(K_X)|_F \geq_{\text{num}} 2\sigma^*(K_{F_0}).$$

Now by a similar argument to (‡) in Claim 3.2.1,  $|K_F + \lceil Q \rceil|$  satisfies the condition of Lemma 2.6. Thus  $|K_F + \lceil Q \rceil|$  gives a birational map and so does  $\varphi_5$ .

**b.** Assume  $K_{F_0}^2 = 2$ . As long as  $K_X^3 > 24$ , we have  $\tau_0 > 4$  which means, by Lemma 2.7,

$$\pi^*(K_X) \geq_{\text{num}} (4 + \delta)F$$

for some very small rational number  $\delta > 0$  and

$$\pi^*(K_X)|_F \geq_{\text{num}} \left(\frac{4}{5} + \varepsilon_0\right)\sigma^*(K_{F_0})$$

for some small rational number  $\varepsilon_0 > 0$ . The vanishing theorem gives

$$|K_{X'} + \lceil 4\pi^*(K_X) \rceil|_F \supset |K_F + 3\sigma^*(K_{F_0}) + \lceil Q_b \rceil| + (\text{fixed divisor})$$

where  $Q_b$  is certain nef and big  $\mathbb{Q}$ -divisor. Now an easy exercise applying the vanishing theorem on surfaces shows that  $K_F + \sigma^*(K_{F_0}) + \lceil Q_b \rceil$  is effective due to the fact that  $\sigma^*(K_{F_0})$  is 1-connected. This means that  $\varphi_5|_F$  distinguishes different general curves  $C$  in  $|G| := |2\sigma^*(K_{F_0})|$ . Finally one sees  $\varphi_5$  is birational by [9, Theorem 3.6].

**c.** Assume  $K_{F_0}^2 = 1$ . As long as  $K_X^3 > 12$ , we have  $\tau_0 > 4$  and then, similarly, we are reduced to prove that  $|K_F + 3\sigma^*(K_{F_0}) + \lceil Q_c \rceil|$  gives a birational map where  $Q_c$  is certain nef and big  $\mathbb{Q}$ -divisor. This is the case, according to [9, Theorem 3.6], by taking  $|G| := |2\sigma^*(K_{F_0})|$  if  $p_g(F) = 1$  and  $|G|$  to be the moving part of  $|K_F|$  if  $p_g(F) = 2$ . We also leave this as an easy exercise.

In a word,  $\varphi_5$  is birational when  $K_X^3 > 81$  and  $p_g(X) = 2$ .  $\square$

We have proved the following:

**Theorem 4.4.** *Let  $X$  be a minimal projective 3-fold of general type. Assume  $p_g(X) = 2$  and  $K_X^3 > 81$ . Then  $\varphi_5$  is birational.*

Theorem 4.1, Theorem 4.3 and Theorem 4.4 imply Theorem 1.1(3).

## 5. Characterizing the birationality of $\varphi_4$ (Part A)

By Chen-Zhang [9, Theorem 1.3], we only need to study the cases  $p_g(X) = 2, 3, 4$ .



5.1.  $\varphi_4$  in the case  $p_g(X) = 4$  and  $d_1 = 3$ .

Keep the same setting and notation as in 2.1. Pick a general member  $S \in |M_1|$ . Consider the linear system  $|4K_{X'}|$  and its sub-system  $|K_{X'} + \lceil 2\pi^*(K_X) \rceil + M_1|$ . Kawamata-Viehweg vanishing gives the relation

$$\begin{aligned} |K_{X'} + \lceil 2\pi^*(K_X) \rceil + M_1|_S &= |K_S + \lceil 2\pi^*(K_X) \rceil|_S \\ &\supset |K_S + \lceil 2L \rceil| + (\text{fixed divisor}) \end{aligned} \quad (5.1)$$

where  $L := \pi^*(K_X)|_S$  is an effective nef and big  $\mathbb{Q}$ -divisor on  $S$ . Set  $|G| = |M_1|_S$ . Pick a generic irreducible element  $C$  of  $|G|$ . Then, since  $p_g(S) > 0$ ,  $|K_S + \lceil 2L \rceil|$  distinguishes different general curves  $C$ . So it suffices to prove the birationality (or non-birationality) of  $\varphi_4|_C$ . In fact, Kawamata-Viehweg vanishing gives, furthermore,

$$|K_F + \lceil 2L - E'_1 \rceil|_F|_C = |K_C + D_3|$$

where  $D_3 := \lceil 2L - E'_1 \rceil_F - C|_C$  with  $\deg(D_3) \geq \xi := (L \cdot C)$ .

**Claim 5.1.1.**  $K_X^3 > 2$  if and only if  $\xi > 2$ .

*Proof.* Pick a general surface  $S \in |M_1|$ . We have

$$\pi^*(K_X)|_S \sim S|_S + E'_1|_S$$

and so

$$K_X^3 = (\pi^*(K_X))^3 \geq (\pi^*(K_X)^2 \cdot S) = \xi.$$

On  $S$ , since  $|C|$  is not composed of a pencil of curves,  $C^2 \geq 2$ . Thus  $\xi = (\pi^*(K_X) \cdot S^2) \geq C^2 \geq 2$ .

On the other hand, by choosing a sufficiently large and divisible integer  $n$  to make  $|n\pi^*(K_X)|$  base point free, one can apply the Hodge index theorem on the smooth surface  $S_{[n]} \in |n\pi^*(K_X)|$  to get the inequality:

$$\xi = (\pi^*(K_X) \cdot S^2) = \frac{1}{n}(\pi^*(K_X)|_{S_{[n]}} \cdot S|_{S_{[n]}}) \geq \sqrt{K_X^3 \cdot \xi}.$$

By [8, Theorem 1.5(2)], we have  $K_X^3 \geq 2$ . Thus it follows that  $\xi = 2$  if and only  $K_X^3 = 2$ . The lemma is proved.  $\square$

**Claim 5.1.2.**  $\varphi_4$  is generically finite of degree  $\leq 2$ .

*Proof.* By definition,  $p = 1$  and  $\beta = 1$ . Then  $\alpha_4 = \xi$ . Whenever  $\xi > 2$ , [9, Theorem 3.6] implies the birationality of  $\varphi_4$ . Otherwise,  $\xi = 2$  implies  $K_X^3 = 2$  and since we have

$$K_X^3 \geq S^3 \geq \deg(\varphi_1) \geq 2, \quad (5.2)$$

$\deg(\varphi_1) = 2$ , i.e.  $\varphi_1$  must be generically finite of degree 2.  $\square$

**Claim 5.1.3.** When  $K_X^3 = 2$ ,  $\varphi_4$  is generically finite of degree 2.

*Proof.* As we have seen in the previous Claim,  $\varphi_1$  is generically finite of degree 2. This means  $\varphi_1|_C$  is a double cover onto  $\mathbb{P}^1$ . In particular,  $C$  is hyperelliptic and  $M_1|_C$  is exactly a  $g_2^1$  of  $C$ . Note that  $C$  is a curve of genus  $\geq 4$  since  $(K_S \cdot C) + C^2 \geq 6$ . We have

$$\begin{aligned} |K_F + 2L|_C &\supset |K_F + 2S|_S|_C + (\text{fixed divisor}) \\ &= |K_C + S|_C| + (\text{fixed divisor}) \end{aligned}$$

by the vanishing theorem. This, together with the relation (5.1), implies  $|M_4|_C \supset |K_C + S|_C| + (\text{fixed divisor})$ , where the last one is base point free with  $\deg(K_C + S|_C) \geq 8$ . Since  $(4\pi^*(K_X) \cdot C) = 8$ , we see  $|M_4|_C = |K_C + S|_C|$ , which gives a double cover. Thus  $\varphi_4$  is generically a double cover.  $\square$

Claim 5.1.1, Claim 5.1.2 and Claim 5.1.3 directly imply the following:

**Theorem 5.1.** *Let  $X$  be a minimal projective 3-fold of general type. Assume  $p_g(X) = 4$  and  $\varphi_1$  is generically finite. Then  $\varphi_4$  is birational if and only if  $K_X^3 = 2$ .*

## 5.2. $\varphi_4$ in the case $p_g(X) = 4$ and $d_1 = 2$ .

In this case,  $\dim(\Gamma) = \dim(X) - 1 = 2$ . Pick a general fiber  $C$  of  $f$ . We have

$$\pi^*(K_X)|_S \equiv w_2 C + E'_1|_S$$

where

$$w_2 = \deg(s) \deg(\varphi_1(X')) \geq p_g(X) - 2 = 2. \quad (5.3)$$

Similar to the argument in the last section, we only need to study the property of  $\varphi_4|_C$  for a general curve  $C$ .

**Claim 5.2.1.** *If  $g(C) \geq 3$ ,  $\varphi_4$  is birational.*

*Proof.* We take  $|G| = |S|_S|$  on  $S$ . Then  $\beta = w_2 \geq 2$ . It follows, from [9, Theorem 3.6], that  $\xi \geq \frac{\deg(K_C)}{1 + \frac{1}{\beta} + \frac{1}{\beta}} \geq \frac{8}{5}$ . Then  $\alpha_4 = (4 - 1 - 1 - \frac{1}{2})\xi \geq \frac{12}{5} > 2$ . By [9, Theorem 3.6],  $\varphi_4$  is birational.  $\square$

In the case  $g(C) = 2$ , one gets  $\xi \geq \frac{2}{1 + 1 + \frac{1}{2}} = \frac{4}{5}$ . Then  $\alpha_4 = (4 - 1 - 1 - \frac{1}{2})\xi \geq \frac{6}{5} > 1$ . By [9, Theorem 3.6] once more, one gets  $\xi \geq 1$ .

**Claim 5.2.2.** *Assume  $g(C) = 2$  and  $\xi > 1$ . Then  $\xi$  has the following explicit lower bounds:*

- (A) When  $w_2 \geq 3$ ,  $\xi \geq \frac{6}{5}$ ;  $\xi = \frac{6}{5}$  implies  $w_2 = 3$ ;  $K_X^3 > \frac{72}{5}$  implies  $\xi > \frac{6}{5}$ .
- (B) When  $w_2 = 2$ ,  $\xi \geq \frac{8}{7}$ .

*Proof.* We only prove (A) while omitting parallel argument for (B).

Find an integer  $l_0 > 5$  such that  $\xi \geq \frac{l_0+1}{l_0}$ . Set  $m' = l_0 - 1$  and then we have

$$\alpha_{m'} = (l_0 - 1 - 2 - \frac{1}{\beta})\xi \geq (l_0 - \frac{10}{3}) \cdot \frac{l_0 + 1}{l_0} > l_0 - 3 > 1.$$

By [9, Theorem 3.6], one gets  $\xi \geq \frac{l_0}{l_0-1}$ . Recursively running this program as long as  $m' \geq 5$ , so we eventually get  $\xi \geq \frac{6}{5}$ . Clearly, if  $w_2 > 3$ , one gets  $\xi > \frac{6}{5}$ .

If  $K_X^3 > \frac{72}{5}$ , since  $(\pi^*(K_X) \cdot S^2) \geq 3\xi = \frac{18}{5}$ , we have

$$(\pi^*(K_X)|_S)^2 \geq \sqrt{K_X^3 \cdot (\pi^*(K_X) \cdot S^2)} \geq \sqrt{\frac{18}{5} K_X^3}$$

by the Hodge index theorem on a general member of  $|n\pi^*(K_X)|$ . Suppose  $\xi = \frac{6}{5}$ . Since  $\nu_0 = \frac{(\pi^*(K_X)|_S)^2}{2\xi} > 3$ , Lemma 2.8 implies

$$\pi^*(K_X)|_F \geq_{\text{num}} (3 + \eta)C$$

for a small rational number  $\eta > 0$ . Now with  $\beta > 3$  and applying [9, Theorem 3.6] one more time, one would get  $\xi > \frac{6}{5}$ , a contradiction. Thus anyway we have  $\xi > \frac{6}{5}$ . We are done.  $\square$

**Claim 5.2.3.** *Assume  $g(C) = 2$ . Then  $\varphi_4$  is birational in any of the following cases:*

- (a)  $w_2 \geq 3$  and  $\xi > \frac{6}{5}$ ;
- (b)  $w_2 = 2$  and  $\xi > \frac{4}{3}$ .

*Proof.* For case (a), since  $\alpha_4 = (4 - 1 - 1 - \frac{1}{\beta})\xi \geq \frac{5}{3}\xi > 2$ , [9, Theorem 3.6] implies that  $\varphi_4$  is birational.

For case (b), since  $\alpha_4 \geq \frac{3}{2}\xi > 2$ ,  $\varphi_4$  is birational.  $\square$

**Claim 5.2.4.** *When  $g(C) = 2$ ,  $w_2 = 2$  and  $\frac{8}{7} \leq \xi \leq \frac{4}{3}$ ,  $\varphi_4$  is birational.*

*Proof.* We consider the surface  $\Sigma := \varphi_1(X') \subset \mathbb{P}^3$ . By the inequality (5.3), we have  $\deg(\Sigma) = 2$ . Classical surface theory (cf. Reid [20, Ex.19 at p30]) says that  $\Sigma$  must be one of the following surfaces:

- (I)  $\Sigma$  is the cone  $\bar{\mathbb{F}}_2$  obtained by blowing-down the unique section of self-intersection number  $(-2)$  on Hirzebruch surface  $\mathbb{F}_2$  (a relatively minimal ruled surface).
- (II)  $\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$ .

In both cases,  $\Sigma$  is normal. By our choice of  $\pi$ , we may assume that  $\Gamma$  is over the resolution of singularities (if any) of  $\bar{\mathbb{F}}_2$ , i.e.  $\Gamma$  is over  $\mathbb{F}_2$  in the first case. By pulling back hyperplane sections from  $\Sigma$  to  $\Gamma$ , we have a base point free divisor  $H_\Gamma = s^*(\mathcal{O}_\Sigma(1))$  such that  $M_1 \sim f^*(H_\Gamma)$ . We now analyze the structure of  $H_\Gamma$  case by case.

Case (I). Denote by  $\nu : \mathbb{F}_2 \rightarrow \Sigma$  the blow up at the singularity of  $\bar{\mathbb{F}}_2$ . Denote  $H_2 := \nu^*(\mathcal{O}_\Sigma(1))$ . Then  $h^0(\mathbb{F}_2, H_2) = 4$ . Noting that  $H_2$  is a nef and big divisor on  $\mathbb{F}_2$ , we can write

$$H_2 \sim \mu G_0 + n\mathcal{T}$$

where  $G_0$  is the unique section of the ruling structure with  $G_0^2 = -2$ ,  $\mathcal{T}$  is the general fiber of the ruling of  $\mathbb{F}_2$ ,  $\mu$  and  $n$  are integers. Necessarily we get  $n = 2$  and  $\mu = 1$ . Furthermore  $G_0 + \mathcal{T}$  is nef. Let  $\theta_0 : \mathbb{F}_2 \rightarrow \mathbb{P}^1$  be the  $\mathbb{P}^1$ -bundle fibration and  $\eta_2 : \Gamma \rightarrow \mathbb{F}_2$  the birational morphism. Let  $f_0 : X' \rightarrow \mathbb{P}^1$  be the composition, i.e.  $f_0 := \theta_0 \circ \eta_2 \circ f$ . Let  $\hat{F}$  be a general fiber of  $f_0$ . Clearly, we see  $S \sim M_1 \geq 2\hat{F} + N_0$  where  $N_0 = f^*\eta_2^*(G_0)$ . Since  $G_0 + \mathcal{T}$  is nef,  $N_0 + \hat{F}$  is nef. Observing that  $|S|_S$  is composed of a pencil of curves,  $|\hat{F}|_S$  must be a sub-pencil of  $|S|_S$  which means  $\hat{F}|_S \equiv uC$  for some integer  $u > 0$ . In particular,  $S|_{\hat{F}} \geq C$  and, for similar reason,  $N_0|_{\hat{F}} \geq C$  since  $G_0 \cap \mathcal{T}$  is a point on  $\mathcal{T}$ . Now Kawamata-Viehweg vanishing tells

$$\begin{aligned} & |K_{X'} + \lceil 2\pi^*(K_X) \rceil + 2\hat{F} + N_0|_{\hat{F}} \\ &= |K_{\hat{F}} + \lceil 2\pi^*(K_X) \rceil|_{\hat{F}} + N_0|_{\hat{F}} \\ &\supset |K_{\hat{F}} + \lceil 2\pi^*(K_X) \rceil|_{\hat{F}} + C| + (\text{fixed divisor}). \end{aligned}$$

Applying the vanishing again, we get

$$|K_{\hat{F}} + \lceil 2\pi^*(K_X) \rceil|_{\hat{F}} + C|_C = |K_C + \hat{D}|$$

with  $\deg(\hat{D}) \geq 2\xi > 2$ . Noting that

$$|K_{X'} + \lceil 2\pi^*(K_X) \rceil + 2\hat{F} + N_0| + (\text{fixed divisor}) \subset |4K_{X'}|,$$

we see that  $\varphi_4$  distinguishes different general surfaces  $\hat{F}$ , different general curves  $C$  and that  $\varphi_4$  separates different general points on  $C$ . This shows that  $\varphi_4$  is birational by the birationality principle.

Case (II). We just consider the morphism  $g := s \circ f : X' \rightarrow \Sigma$ . Since  $\mathcal{O}_\Sigma(1) \sim L_1 + L_2$  with  $(L_1 \cdot L_2) = 1$ , the pull backs of  $L_1$  and  $L_2$  form two fiber structures on  $X'$ . Set  $F_1 := g^*(L_1)$  and  $F_2 := g^*(L_2)$ . Then  $S \geq F_1 + F_2$ . We may assume that both  $F_1$  and  $F_2$  are irreducible for general  $L_1$  and general  $L_2$ . Otherwise, we are in much better situation. Similarly the vanishing theorem gives

$$|K_{X'} + \lceil 2\pi^*(K_X) \rceil + F_1 + F_2|_{F_1} \supset |K_{F_1} + \lceil 2\pi^*(K_X) \rceil|_{F_1} + C| + (\text{fixed divisor})$$

and

$$|K_{F_1} + \lceil 2\pi^*(K_X) \rceil|_{F_1} + C|_C = |K_C + \tilde{D}_1|$$

with  $\deg(\tilde{D}_1) \geq 2\xi > 2$ . This also implies the birationality of  $\varphi_4$ . We are done.  $\square$

**Lemma 5.2.** *When  $p_g(X) \geq 3$ ,  $d_1 = 2$  and  $\xi = 1$ ,  $\varphi_4$  is non-birational.*

*Proof.* This is simply a copy of [9, Proposition 4.6] where the proof trivially follows with  $p_g(X) = 3, 4$ . So we omit the details.  $\square$

So we can conclude the following:

**Theorem 5.3.** *Let  $X$  be a minimal projective 3-fold of general type. Assume  $p_g(X) = 4$ ,  $K_X^3 > \frac{72}{5}$  and  $d_1 = 2$ . Then  $\varphi_4$  is non-birational if and only if  $(K_X \cdot C) = 1$  for the general fiber  $C$  of  $f$ . In this situation,  $g(C) = 2$ .*

**Remark 5.4.** We put on the assumption  $K_X^3 > \frac{72}{5}$  in Theorem 5.3 just to eliminate the situation:  $g(C) = 2$ ,  $w_2 = 3$  and  $\xi = \frac{6}{5}$ . We do not know if this can really happen. It might be possible to analyze it in a similar way to that of Claim 5.2.4. But then, since the surface  $\Sigma$  is of degree 3 in  $\mathbb{P}^3$ , there would be fairly many cases to do. Such a surface  $\Sigma$  can be either normal or non-normal (see, for instance, Abe-Furushima [1], Miyanishi-Zhang [15, 16], Reid [18] and Ye [23]).

### 5.3. $\varphi_4$ in the case $p_g(X) = 4$ and $d_1 = 1$ .

We have an induced fibration  $f : X' \rightarrow \Gamma$  with the general fiber  $F$ . Since  $p_g(F) > 0$  and by the surface theory,  $F$  must be among the following types:

- (a)  $p_g(F) = 2$  and  $K_{F_0}^2 = 1$ .
- (b)  $K_{F_0}^2 \geq 3$ .
- (c) other surfaces with  $p_g(F) > 0$ .

**Claim 5.3.1.** *If  $F$  is of Type (a),  $\varphi_4$  is automatically non-birational and  $X$  has a natural curve family  $\mathcal{C}$  with  $(K_X \cdot C_0) = 1$  for the general member  $C_0 \in \mathcal{C}$ .*

*Proof.* Just take the relative canonical map of  $f$  as what we have seen in [9, Theorem 1.4]. The curve family  $\mathcal{C}$  is composed of all those fibers of the relative canonical map of  $f$ . We omit more details to avoid unnecessary redundancy.  $\square$

**Claim 5.3.2.** *If  $F$  is of Type (b), then  $\varphi_4$  is birational.*

*Proof.* According to [9, 4.8], it is sufficient to assume  $g(\Gamma) = 0$ , i.e.  $\Gamma \cong \mathbb{P}^1$ . Pick a general fiber  $F$  of  $f$ . By Kawamata-Viehweg vanishing, we have

$$|K_{X'} + [3\pi^*(K_X) - \frac{1}{3}E'_1]|_F \supset |K_F + [L_{1/3}]| + (\text{fixed divisor}) \quad (5.4)$$

where  $L_{1/3} := (3\pi^*(K_X) - F - \frac{1}{3}E'_1)|_F \equiv \frac{8}{3}\pi^*(K_X)|_F$ .

By [9, Lemma 3.7], we have

$$L_{1/3} \geq_{\text{num}} (2 - \varepsilon)\sigma^*(K_{F_0})$$

for any small rational number  $\varepsilon > 0$ . Since  $K_{F_0}^2 \geq 3$ , a similar argument to (‡) in Claim 3.2.1 shows that  $|K_F + [L_{1/3}]|$  satisfies the conditions of Lemma 2.6. Thus it follows that  $\varphi_4$  is birational.  $\square$

**Claim 5.3.3.** *If  $F$  is of Type (c) and  $K_X^3 > 12$ ,  $\varphi_4$  is birational.*

*Proof.* Take  $|G| := |2\sigma^*(K_{F_0})|$ . Lemma 2.9 says that  $\varphi_4$  distinguishes different general members of  $|G|$ . Since, under the condition of the claim, one has  $\tau_0 > 3$ , Lemma 2.7 implies

$$\pi^*(K_X) \geq_{\text{num}} (3 + \delta_0)F$$

for some very small rational number  $\delta_0 > 0$  and

$$L_{1/3} \equiv \frac{8}{3}\pi^*(K_X)|_F \geq_{\text{num}} (2 + \eta_0)\sigma^*(K_{F_0})$$

which directly implies the birationality of  $\varphi_4|_F$  by [9, Theorem 3.6]. We are done.  $\square$

So we have proved the following:

**Theorem 5.5.** *Let  $X$  be a minimal projective 3-fold of general type. Assume  $p_g(X) = 4$ ,  $K_X^3 > 12$  and  $d_1 = 1$ . Then  $\varphi_4$  is birational if and only if  $X$  is not canonically fibred by (1,2) surfaces.*

Now we prove the following:

**Theorem 5.6.** *(=Theorem 1.2(1)) Let  $X$  be a minimal projective 3-fold of general type. Assume  $p_g(X) = 4$  and  $K_X^3 > \frac{72}{5}$ . Then  $\varphi_4$  is birational if and only if  $X$  does not contain any genus 2 curve family of canonical degree 1.*

*Proof.* If  $\varphi_4$  is non-birational, then Theorem 5.1, Theorem 5.3 and Theorem 5.5 imply that  $X$  is either canonically fibred by curves of canonical degree 1 or canonically fibred by (1,2) surfaces. For the later case, there is also a genus 2 curve family of canonical degree 1 by Claim 5.3.1.

Conversely, if  $X$  admits a genus 2 curve family  $\mathcal{C}$  of canonical degree 1, we see  $d_1 \leq 2$ . Otherwise,  $(K_X \cdot C_0) = (\pi^*(K_X) \cdot \tilde{C}_0) \geq (S \cdot \tilde{C}_0) \geq 2$  for a general member  $C_0 \in \mathcal{C}$  since  $|S|_{\tilde{C}_0}$  gives a generically finite map, where  $\tilde{C}_0$  denotes the strict transform of  $C_0$ .

Now assume  $d_1 = 2$ . We want to show that  $\mathcal{C}$  is the canonical curve family, i.e.  $C = \tilde{C}_0$ . Otherwise,  $(\pi^*(K_X) \cdot \tilde{C}_0) \geq (S \cdot \tilde{C}_0) \geq 2$  since  $|S|_{\tilde{C}_0}$  is moving and  $g(\tilde{C}_0) = 2$ , a contradiction. Now since  $(\pi^*(K_X) \cdot C) = 1$ ,  $\varphi_4$  is non-birational by Theorem 5.3.

Assume  $d_1 = 1$ . If  $\tilde{C}_0 \not\subset F$  for a general fiber  $F$ , then  $F|_{\tilde{C}_0}$  is moving and thus  $(\pi^*(K_X) \cdot \tilde{C}_0) \geq (F \cdot \tilde{C}_0) \geq 2$ , a contradiction. Thus  $\tilde{C}_0 \subset F$ . If  $F$  is not a (1,2) surface, we have

$$(\pi^*(K_X) \cdot \tilde{C}_0) \geq \frac{2}{3}(\sigma^*(K_{F_0}) \cdot \tilde{C}_0) \geq \frac{4}{3}$$

since  $\tilde{C}_0$  is moving on  $F$ , which is again impossible. So  $F$  is a (1,2) surface. Clearly  $\varphi_4$  is non-birational.  $\square$

**Remark 5.7.** When  $\varphi_4$  is non-birational, the curve family in Theorem 1.2(1) is uniquely determined by  $X$ . Such family is called “canonical curve family” of  $X$ .

## 6. Characterizing the birationality of $\varphi_4$ (Part B)

We study  $\varphi_4$  for the cases  $p_g(X) = 2, 3$  in this section.

### 6.1. $\varphi_4$ in the case $p_g(X) = 3$ and $d_1 = 2$ .

This is corresponding to Subsection 4.1. We keep the same notation there.

**Claim 6.1.1.** *If  $K_X^3 > 48$  and  $g(C) \geq 3$ ,  $\varphi_4$  is birational.*

*Proof.* We have seen from Claim 4.1.1 that  $\xi \geq \frac{4}{3}$ .

If  $\xi > 2$ , then  $|K_F + \lceil 2L \rceil|_C \supset |K_C + \lceil L \rceil|_C + (\text{fixed divisor})$  and the last linear system gives a birational map. Thus  $\varphi_4$  is birational.

Assume  $\xi \leq 2$ . Then since

$$L^2 = (\pi^*(K_X)^2 \cdot S) \geq \sqrt{K_X^3 \cdot (\pi^*(K_X) \cdot S)^2} \geq \sqrt{\frac{4}{3} K_X^3},$$

we have  $\nu_0 > 2$  and Lemma 2.8 implies

$$\pi^*(K_X)|_S \equiv (2 + \eta_0)C + J_{\eta_0}$$

for some very small rational number  $\eta_0 > 0$ . Now the vanishing theorem gives

$$|K_F + \lceil 2L - \frac{1}{2 + \eta_0} J_{\eta_0} \rceil|_C = |K_C + D_{\eta_0}|$$

where  $D_{\eta_0} := \lceil 2L - C - \frac{1}{2 + \eta_0} J_{\eta_0} \rceil|_C$  with

$$\deg(D_{\eta_0}) \geq (2 - \frac{1}{2 + \eta_0})\xi > 2.$$

Thus  $\varphi_4|_C$  is birational and so is  $\varphi_4$ .  $\square$

**Claim 6.1.2.** *If  $K_X^3 > 144$ ,  $g(C) = 2$  and  $\xi > 1$ , then  $\varphi_4$  is birational.*

*Proof.* One has  $(\pi^*(K_X) \cdot S^2) \geq \xi > 1$ . On the other hand, since  $\xi \leq (K_F \cdot C) = 2$ , one has  $\nu_0 \geq \frac{(\pi^*(K_X)^2 \cdot S)}{4} > 3$  and Lemma 2.8 implies

$$\pi^*(K_X)|_S \equiv (3 + \eta_1)C + J_{\eta_1}$$

for some rational number  $\eta_1 > 0$  and for some effective  $\mathbb{Q}$ -divisor  $J_{\eta_1}$ . In particular, one has  $\beta > 3$ .

Find an integer  $l_0 > 5$  such that  $\xi \geq \frac{l_0 + 1}{l_0}$ . Set  $m' = l_0 - 1$  and then we have

$$\alpha_{m'} = (l_0 - 1 - 2 - \frac{1}{\beta})\xi > (l_0 - \frac{10}{3}) \cdot \frac{l_0 + 1}{l_0} > l_0 - 3 > 1.$$

By [9, Theorem 3.6], one gets  $\xi \geq \frac{l_0}{l_0 - 1}$ . Recursively running this program as long as  $m' \geq 5$ , so we eventually get  $\xi \geq \frac{6}{5}$ .

Now the vanishing theorem gives

$$|K_F + \lceil 2L - \frac{1}{3 + \eta_1} J_{\eta_1} \rceil|_C = |K_C + D_{\eta_1}|$$

where  $D_{\eta_1} := \lceil 2L - C - \frac{1}{2+\eta_1} J_{\eta_1} \rceil|_C$  with  $\deg(D_{\eta_1}) \geq (2 - \frac{1}{3+\eta_1})\xi > 2$ . Thus  $\varphi_4|_C$  is birational and so is  $\varphi_4$ .  $\square$

## 6.2. $\varphi_4$ in the case $p_g(X) = 2, 3$ and $d_1 = 1$ .

**Claim 6.2.1.** *If  $K_X^3 > 180$ ,  $p_g(X) = 3$  and  $F$  is not a  $(1,2)$  surface,  $\varphi_4$  is birational.*

*Proof.* We organize the proof by analyzing the numerical types of  $F$ .

If  $K_{F_0}^2 \geq 19$ , we have seen  $p_g(F) \geq 2$  in the proof of Claim 4.3.1. Take  $|G|$  to be the moving part of  $|K_F|$ . Then  $\beta \mapsto \frac{2}{3}$  by [9, Lemma 3.7]. Pick a generic irreducible element  $C$  of  $|G|$ . Clearly  $\varphi_4|_F$  distinguishes different general curves  $C$ . When  $|G|$  is not composed of a pencil, then  $C^2 \geq 2$  and  $(\sigma^*(K_{F_0}) \cdot C) \geq \sqrt{2K_{F_0}^2} \geq \sqrt{38}$ . Thus  $\xi \geq \frac{2}{3}(\sigma^*(K_{F_0}) \cdot C) > 2$ . Since  $\alpha_4 \geq (4 - 1 - \frac{1}{2} - \frac{1}{\beta})\xi > 2$ ,  $\varphi_4$  is birational. When  $|G|$  is composed of a pencil and  $\xi > 2$ ,  $\varphi_4$  is birational for the same reason. Suppose  $\xi \leq 2$ . Since we have  $(\pi^*(K_X)|_F)^2 \geq \frac{4}{9}K_{F_0}^2$  and  $K_{F_0}^2 \geq 19$ , Lemma 2.8 implies  $\nu_0 > 2$  and thus  $\beta > 2$ . Noting that  $\xi \geq \frac{2}{3}(\sigma^*(K_{F_0}) \cdot C) \geq \frac{4}{3}$  since  $F$  is not a  $(1,2)$  surface, we have  $\alpha_4 > (4 - 1 - \frac{1}{2} - \frac{1}{\beta})\xi \geq 2$  and thus  $\varphi_4$  is birational.

If  $K_{F_0}^2 \leq 18$  and  $K_X^3 > 180$ , Lemma 2.7 implies  $\pi^*(K_X) \geq_{\text{num}} \frac{10}{3}F$  and  $\pi^*(K_X)|_F \geq \frac{10}{13}\sigma^*(K_{F_0})$ . Take  $|G|$  to be the moving part of  $|K_F|$  whenever  $F$  is a  $(2,3)$  surface and, otherwise,  $|G| := |2\sigma^*(K_{F_0})|$ . By Lemma 2.9,  $\varphi_4|_F$  distinguishes different generic irreducible elements  $C$  of  $|G|$ . We have  $\alpha_4 > 2$  in the  $(2,3)$  case and  $\alpha_4 > 0$  otherwise and thus  $\varphi_4$  is birational by [9, Theorem 3.6].  $\square$

So we can conclude the following:

**Theorem 6.1.** *(=Theorem 1.2(2)) Let  $X$  be a minimal projective 3-fold of general type. Assume  $p_g(X) = 3$  and  $K_X^3 > 180$ . Then  $\varphi_4$  is birational if and only if  $X$  does not contain any genus 2 curve family of canonical degree 1.*

*Proof.* This is parallel to Theorem 5.6 by virtue of Claim 6.1.1, Claim 6.1.2 and Claim 6.2.1. We omit the details.  $\square$

Finally we prove the following:

**Theorem 6.2.** *(=Theorem 1.2(3)) Let  $X$  be a minimal projective 3-fold of general type. Assume  $p_g(X) = 2$  and  $K_X^3 > 855$ . Then  $\varphi_4$  is birational if and only if  $X$  does not contain any genus 2 curve family of canonical degree 1.*

*Proof.* Suppose  $X$  does not contain any genus 2 curve family of canonical degree 1. We want to show  $\varphi_4$  is birational. We discuss this according to the numerical types of  $F$  while we have an induced fibration  $f : X' \rightarrow \Gamma$ . When  $g(\Gamma) > 0$ , since  $F$  is not a  $(1,2)$  surface (otherwise,



$X$  will have a genus 2 curve family of canonical degree 1), we have known from [9] that  $\varphi_4$  is birational. So we may assume  $\Gamma \cong \mathbb{P}^1$ .

Assume  $K_{F_0}^2 > 96$ . It is sufficient to show that  $|K_F + \lceil 2L \rceil|$  gives a birational map where  $L := \pi^*(K_X)|_F$ . We have

$$(2L)^2 \geq K_{F_0}^2 > 8$$

by assumption. Pick two distinct points  $P_1, P_2$  in very general positions of  $F$ . If all the curves  $C_{1,2}$  passing through  $P_1$  and  $P_2$  satisfy  $(\sigma^*(K_{F_0}) \cdot C_{1,2}) \geq 4$  (which means  $(2L \cdot C_{1,2}) \geq 4$ ), then Lemma 2.6 implies that  $|K_F + \lceil 2L \rceil|$  separates  $P_1$  and  $P_2$ . Otherwise, there is such a curve  $C_{1,2}$  with  $(\sigma^*(K_{F_0}) \cdot C_{1,2}) < 4$  and, in fact, these curves  $C_{1,2}$  form a curve family (since  $P_1$  and  $P_2$  are in very general positions). Thus, by our assumption, we have  $(L \cdot C_{1,2}) > 1$ . Since  $K_{F_0}^2 > 96$ , we have

$$\nu_0 \geq \frac{L^2}{8} \geq \frac{1}{32} K_{F_0}^2 > 3.$$

Lemma 2.8 implies  $\beta_{1,2} > 3$ . We may estimate the intersection number  $\xi_{1,2} := (L \cdot C_{1,2})$  using similar method to that in the proof of Claim 6.1.2. The result is  $\xi_{1,2} \geq \frac{6}{5}$ . Now we have  $\alpha_4^{1,2} \geq (4 - 1 - 1 - \frac{1}{\beta_{1,2}})\xi > 2$  and [9, Theorem 3.6] implies that  $\varphi_4|_{C_{1,2}}$  is birational and, in particular,  $\varphi_4$  separates  $P_1$  and  $P_2$  who are in very general positions. Thus we have seen  $\varphi_4$  is birational.

Assume  $K_{F_0}^2 \leq 95$  and  $K_X^3 > 855$ . As we have seen before,  $F$  can not be a (1,2) surface. Take  $|G|$  to be the moving part of  $|K_F|$  whenever  $F$  is a (2,3) surface and, otherwise,  $|G| := |2\sigma^*(K_{F_0})|$ . Lemma 2.7 implies  $\tau_0 > 3$  and

$$\pi^*(K_X)|_F \geq_{\text{num}} \left(\frac{3}{4} + \delta_0\right) \sigma^*(K_{F_0})$$

for some small rational number  $\delta_0 > 0$ . By the vanishing theorem, we are in the position to show  $|K_F + 2\sigma^*(K_{F_0}) + \lceil Q_f \rceil|$  gives a birational map, where  $Q_f$  is a nef and big  $\mathbb{Q}$ -divisor. In fact, when  $F$  is a (2,3) surface, this is the case due to [9, Theorem 3.6]. When  $F$  is neither a (2,3) surface nor a (1,1) surface,  $|K_F + 2\sigma^*(K_{F_0}) + \lceil Q_f \rceil|$  satisfies the condition of Lemma 2.6 by a parallel argument to (‡) in the proof of Claim 3.2.1 and thus it also gives a birational map. Finally if  $F$  is a (1,1) surface, Lemma 2.7 actually gives  $\tau_0 > 200$  and we are in a much better situation. The vanishing theorem again allows us to consider the linear system  $|K_F + \lceil Q_{1,1} \rceil|$  where  $Q_{1,1} \geq_{\text{num}} (2 + \frac{199}{201})\sigma^*(K_{F_0})$  and  $Q_{1,1}$  is nef and big. Clearly, this linear system also satisfies the condition of Lemma 2.6 still by a similar argument to (‡). In a word,  $\varphi_4$  is birational.

Conversely, if  $X$  has a genus 2 curve family  $\mathcal{C}$  of canonical degree 1,  $\varphi_4$  is non-birational. This can be seen by a similar argument to that of [9, Proposition 4.6]. The point is that, while  $\pi^*(K_X) \cdot C = 1$ , we will be able to see  $|4K_{X'}|_C = |2K_C|$  for a general curve  $C \in \mathcal{C}$  (This is not

a trivial statement at all!). Since  $g(C) = 2$ ,  $\varphi_4$  can not be birational. We omit more details and leave it as an exercise.  $\square$

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Institute of Mathematics & LMNS, Fudan University, Shanghai 200433, China  
E-mail address: mchen@fudan.edu.cn